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UNSTABLE GROUND STATE OF NONLINEAR KLEIN-GORDON EQUATIONS

BY

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ABSTRACT. In this paper we prove the instability of the ground state, i.e. least energy steady-state solution of nonlinear Klein-Gordon equations with space dimension $n \geqslant 3$.

0. Introduction. We present here a proof of instability of the "ground state", the least energy steady-state solution of the equation

$$u_{tt} - \Delta u + f(|u|)\arg u = 0, \qquad x \in \mathbf{R}^n, n \geqslant 3. \tag{0.1}$$

The instability of the ground state is shown for very general nonlinearities f. The only assumptions on f are those that insure the existence of a nontrivial steady state solution of (0.1). If we denote the ground state by u_e, then we have

THEOREM. *Let $u(t)$ be a solution of* (0.1) *such that $u(0) = u_0$, $u_t(0) = u_1$. Then there are initial data (u_0, u_1) arbitrarily close to $(u_e, 0)$ and a sequence (t_k) such that $\|u(t_k)\| \to \infty$ as $k \to \infty$.*

The theory of linearized stability gives a clue to the instability of the ground state since the linearized equation $v_{tt} - \Delta v + f'(u_e)v = 0$ has a negative eigenvalue.

The stability question was first considered by Derrick [**3**] who formally used the linearized stability argument to show that the ground state is unstable. There are several results on blow up for special nonlinearities such as Payne and Sattinger [**6**] and Berestycki and Cazenave [**2**]. However for certain nonlinearities the solutions do not blow up in finite time but exist globally in time. Therefore to solve this problem we are forced to introduce a *new estimate* that shows the instability of the ground state without blow up for all types of nonlinearities.

The existence of the ground state is basically due to Walter Strauss [**10**]. However in the most general setting (and this is the case we are dealing with here) this result is due to H. Berestycki and P. Lions [**12**].

The method we use for obtaining the ground state is that of Clayton Keller who studied the flow of the equation $u_{tt} + u_t - \Delta u + f(u) = 0$ near a stationary state. Keller showed that the stationary state is a saddle point with an infinite dimensional stable manifold and a nonempty finite dimensional unstable manifold.

Finally it is interesting to compare this result with the "degenerate" one-dimensional case. In this case it is well known that there are stable stationary solutions and even solitons, as in the case of the equation $u_{tt} - u_{xx} + \sin u = 0$.

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NOTATION. We employ here the standard notations

$$H_r^1(\mathbf{R}^n) = \Big\{u, \text{ radially symmetric functions on } \mathbf{R}^n,$$

$$\|u\| \equiv \left(\int |\nabla u(x)|^2\,dx + \int |u(x)|^2\,dx\right)^{1/2} < \infty\Big\},$$

$$L_r^p(\mathbf{R}^n) = \Big\{u, \text{ radially symmetric functions on } \mathbf{R}^n,$$

$$|u|_p = \left(\int |u(x)|^p\,dx\right)^{1/p} < \infty\Big\}.$$

$u_r = \partial u/\partial r$ for $u \in H_r^1(\mathbf{R}^n)$, where $r = |x|$, $x \in \mathbf{R}^n$ and $\arg u = u/|u|$.

1. Existence of a nontrivial stationary solution. In this section we show the existence of a nontrivial solution to the problem

$$-\Delta u + f(|u|)\arg u = 0, \qquad x \in \mathbf{R}^n, \quad n \geqslant 3, \tag{1.1}$$

where $f(|u|)$ satisfies the following assumptions:

(i) $f(0) = 0$ and $f'(0) = 1$.

(ii) $\underline{\lim}\, f(|u|)/|u|^l \geqslant 0$ as $|u| \to \infty$, $l = 1 + 4/(n-2)$.

(iii) There is a u such that $G(|u|) < 0$, where $G'(|u|) = f(|u|)$ and $G(0) = 0$.

In this generality this problem was solved by H. Berestycki and P. Lions [**12**]. For the purpose of showing instability we present here a proof by a method of C. Keller [**5**]. First we set up a minimization problem of a functional $J(u)$ restricted to a manifold M and then show that this minimum is a solution of (1.1).

DEFINITION 1.1.

$$J(u) = \frac{1}{2}\int |\nabla u|^2\,dx + \int G(|u|)\,dx,$$

$$K(u) = \frac{n-2}{2}\int |\nabla u|^2\,dx + n\int G(|u|)\,dx,$$

$$M = \{u \in H_r^1 : K(u) = 0, u \neq 0\}.$$

The space $H_r^1(\mathbf{R}^n)$ has the following property.

LEMMA 1.2. *Let $u \in H_r^1(\mathbf{R}^n)$. Then*

$$|u(x)| \leqslant C_n |x|^{(1-n)/2}\|u\| \quad a.e.,$$

where C_n depends only on n, and the inclusion map

$$H_r^1(\mathbf{R}^n) \hookrightarrow L_r^p(\mathbf{R}^n), \qquad 2 < p < 2 + 4/(n-2),$$

is compact.

PROOF. See Strauss [**10**].

LEMMA 1.3. *If $u_0 \in H_r^1(\mathbf{R}^n)$ and $\int G(|u_0|)\,dx < \infty$ is a solution of* (1.1), *then* $K(u_0) = 0$.

PROOF. Let $u_\beta(x) = u_0(x/\beta)$. Then

$$J(u_\beta) = \frac{1}{2}\int |\nabla u_\beta|^2\,dx + \int G(|u_\beta|)\,dx = \frac{\beta^{n-2}}{2}\int |\nabla u_0|^2\,dx + \beta^n\int G(|u_0|)\,dx.$$

Since u_0 is a solution of (1.1), then $\delta J(u_0) = 0$, or $d(J(u_\beta))/d\beta|_{\beta=1} = 0$. But

$$\left.\frac{d(J(u_\beta))}{d\beta}\right|_{\beta=1} = \frac{n-2}{2}\int |\nabla u_0|^2\,dx + n\int G(|u_0|)\,dx.$$

Then $K(u_0) = 0$.

LEMMA 1.4. *M is a C^1 hypersurface in $H_r^1(\mathbf{R}^n)$ bounded away from zero.*

PROOF. Since

$$M = \left\{u : K(u) = \frac{n-2}{2}\int |\nabla u|^2\,dx + n\int G(|u|)\,dx = 0, u \neq 0\right\}$$

and $f = G'$ is continuous, then $K(u)$ is a C^1 functional. Now consider $u_0 \in H_r^1(\mathbf{R}^n)$ such that $\delta K(u_0) = 0$

$$-(n-2)\Delta u_0 + nf(|u_0|)\arg u_0 = 0. \tag{1.2}$$

Then by Lemma 1.3 applied to equation (1.2) we have

$$\frac{(n-2)^2}{2}\int |\nabla u_0|^2\,dx + n^2\int G(|u_0|)\,dx = 0. \tag{1.3}$$

Therefore if $K(u_0) = 0$ we have from (1.3)

$$\frac{(n-2)^2}{2}\int |\nabla u_0|^2\,dx - \frac{n(n-2)}{2}\int |\nabla u_0|^2\,dx = 0$$

and this implies $u_0 = 0$. Hence for $u_0 \in M$, $\delta K(u_0) \neq 0$ and M is a C^1 hypersurface. Finally since $f'(0) = 1$ and $\underline{\lim}\, f(|u|)/|u|^l \geqslant 0$, $l = 1 + 4/(n-2)$, we have $G(|u|) \geqslant |u|^2/4 - C_0|u|^{l+1}$. Therefore

$$K(u) \geqslant \frac{n-2}{2}\int |\nabla u|^2\,dx + \frac{1}{4}\int |u|^2\,dx - C_0\int |u|^{l+1}\,dx.$$

From Sobolev embedding and $n > 2$ we have $K(u) \geqslant \|u\|^2/4 - C_0\|u\|^\alpha$ where $\alpha > 2$. For $0 < \|u\| < \varepsilon$, ε small, we have $K(u) > 0$ and this implies that M is bounded away from zero.

PROPOSITION 1.5 (EXISTENCE OF THE GROUND STATE). *The minimization problem*

$$J_0 \equiv \inf_{u\in M} J(u) = \inf\left\{\frac{1}{n}\int |\nabla u|^2\,dx, K(u) = 0, u \neq 0\right\}$$

which is equal to

$$J_0 = \inf\left\{\frac{1}{n}\int |\nabla u|^2\,dx, K(u) \leqslant 0, u \neq 0\right\}$$

is attained at $u_3 \in M$ ($u_e \in H_r^1(\mathbf{R}^n)$).

PROOF. First we establish the equivalence of the two minimizations. For any $v_0 \in H_r^1(\mathbf{R}^n)$ such that $K(v_0) < 0$ let $v_\beta(x) = v_0(x/\beta)$. Then

$$K(v_\beta) = \frac{\beta^{n-2}(n-2)}{2}\int |\nabla v_0|^2\,dx + \beta^n n\int G(|v_0|)\,dx.$$

Now for $\beta = 1$, $K(v_1) = K(v_0) < 0$ and for β close to zero $K(v_\beta) > 0$ since $n - 2 > 0$. Therefore there is a $\beta_0 \in (0,1)$ such that $K(v_{\beta_0}) = 0$ (i.e. $v_{\beta_0} \in M$). But

$$J(v_{\beta_0}) = \frac{1}{n}\int |\nabla v_{\beta_0}|^2\,dx = \frac{\beta_0^{n-2}}{n}\int |\nabla v_0|^2\,dx < \frac{1}{n}\int |\nabla v_0|^2\,dx$$

since $\beta_0 \in (0,1)$. Consequently the two minimizations are equivalent.

Let $u_j \in H_r^1(\mathbf{R}^n)$ be a minimizing sequence. Then $\int |\nabla u_j|^2\,dx$ is bounded. Since $\underline{\lim}\, f(|u|)/|u|^l \geqslant 0$, there is $C_f > 0$ such that $G(|u|) \geqslant |u|^2/4 - C_f|u|^{l+1}$. Therefore

$$\begin{aligned} 0 \geqslant K(u_j) &= \frac{n-2}{2}\int |\nabla u_j|^2\,dx + n\int G(|u_j|)\,dx \\ &\geqslant \frac{n-2}{2}\int |\nabla u_j|^2\,dx + \frac{n}{4}\int |u_j|^2\,dx - C_f\int |u_j|^{l+1}\,dx. \end{aligned}$$

By the Sobolev embedding $H_r^1(\mathbf{R}^n) \hookrightarrow L_r^{l+1}(\mathbf{R}^n)$ and the boundedness of $\int |\nabla u_j|^2\,dx$ we have $\|u_j\|$ is bounded. Therefore there is a subsequence, also denoted by u_j, such that

$$u_j \overset{w}{\rightharpoonup} u_e \in H_r^1(\mathbf{R}^n)$$

and

$$u_j \to u_e \in L_r^p(\mathbf{R}^n), \qquad 2 < p < 2 + 4/(n-2) \quad \text{(by compactness)}.$$

Now by lower semicontinuity of weak limits and assumption (ii) we have

$$\frac{1}{n}\int |\nabla u_e|^2\,dx \leqslant \underline{\lim}\ \frac{1}{n}\int |\nabla u_j|^2\,dx = J_0$$

and $K(u_e) \leqslant \underline{\lim}\ K(u_j) \leqslant 0$. By the definition of J_0 the above inequalities are equalities and the weak convergence is strong.

THEOREM 1.6.

$$(1.1) \qquad -\Delta u + f(|u|)\arg u = 0$$

has a nontrivial (least energy) solution, namely u_e.

PROOF. Since u_e is a critical point of $J(u)$ restricted to M, by the Lagrange multiplier we have

$$\delta J(u_e) + \eta\delta K(u_e) = 0, \quad \eta \in \mathbf{R},$$

or

$$(1.4) \qquad -\Delta u_e + f(|u_e|)\arg u_e + \eta\big(-(n-2)\Delta u_e + nf(|u_e|)\arg u_e\big) = 0.$$

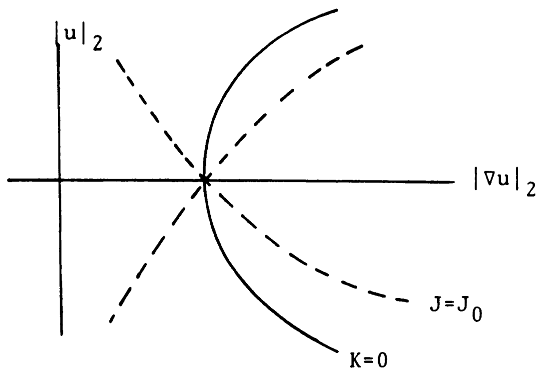

FIGURE 1

By Lemma 1.3 we have

$$K(u_e) + \eta\left[\frac{(n-2)^2}{2}\int |\nabla u_e|^2\,dx + n^2\int G(|u_e|)\,dx\right] = 0$$

or

$$K(u_e) + \eta\left[\frac{(n-2)^2}{2} - \frac{n(n-2)}{2}\int |\nabla u_e|^2\,dx + nK(u_e)\right] = 0. \tag{1.5}$$

But $K(u_e) = 0$ since $u_e \in M$. Hence $\eta\int |\nabla u_e|^2\,dx = 0$ and this implies that $\eta = 0$ and u_e is a solution of (1.1).

REMARK. If we draw the picture in the space $H_r^1(\mathbf{R}^n)$ of $K(u) = 0$ and $J(u) = J_0$ (which is not a manifold) we get formally Figure 1.

2. Instability of the ground state. Consider the Cauchy problem for equation (0.1)

$$u_{tt} - \Delta u + f(|u|)\arg u = 0,$$
$$u(0) = u_0 \in H_r^1(\mathbf{R}^n), \qquad u_t(0) = u_1 \in L_r^2(\mathbf{R}^n).$$

DEFINITION 2.1. Let $X = \{u \in H_r^1;\ \int G(|u|)\,dx < \infty\}$ and define the energy functional for equation (0.1) as

$$E(u,v) = \frac{1}{2}\int |v|^2\,dx + J(u), \qquad u \in X, v \in L_r^2,$$

$$R_1 = \{u \in X, v \in L_r^2\colon E(u,v) < J_0 \text{ and } K(u) > 0\} \cup \{(0,0)\}$$
$$= \left\{u \in X, v \in L_r^2\colon E(u,v) < J_0 \text{ and } \frac{1}{n}\int |\nabla u|^2\,dx < J_0\right\},$$

$$R_2 = \{u \in X, v \in L_r^2\colon E(u,v) < J_0 \text{ and } K(u) < 0, u \neq 0\}$$
$$= \left\{u \in X, v \in L_r^2\colon E(u,v) < J_0 \text{ and } \frac{1}{n}\int |\nabla u|^2\,dx > J_0\right\}.$$

The Cauchy problem does not necessarily have strong solutions $u(\cdot) \in C([0,T), H^1)$ and $u_t(\cdot) \in C([0,T), L^2)$ for the type of nonlinearity we are considering.

DEFINITION. By a solution of (0.1) on the time interval $[0, T)$ we mean a function $u(x, t)$ such that:

(1) u (resp. u_t) is weakly continuous in t on $[0, T)$ with values in $H_r^1(\mathbf{R}^n)$ (resp. $L_r^2(\mathbf{R}^n)$).

(2) $E(u(t), u_t(t)) \leqslant E(u_0, u_1)$.

(3) u satisfies (0.1) in the sense of distributions.

If $u_0 \in X$, $u_1 \in L_r^2$ and f satisfies (ii) of §1, then a solution exists on some time interval $[0, T)$. Moreover if f satisfies the weak hypothesis (basically $|u|f(|u|) \geqslant -C|u|^2$ for u large), then the solutions exist globally in time [**8**]. Uniqueness is also not known for general f [**4**, **8**], neither is regularity when u_0 and u_1 are smooth [**8**].

We multiply equation (0.1) by $\phi_m(r, t)\bar{u}_r$, where

$$\phi_m(r, t) = rH(m - r) + (\ln(m)r/\ln(r))(H(r - m) - H(r - (t + 2m))) + (\ln(m)(t + 2m)/\ln(t + 2m))H(r - (t + 2m))$$

and $H(s) = 1$, $s \geqslant 0$, $H(s) = 0$, $s < 0$, ln is the natural log function, and take the real part

$$\begin{aligned}(2.1)\quad \frac{d}{dt}\int \operatorname{Re}\phi_m\bar{u}_r u_t\,dx &+ \int_{r<m}\left(n|u_t|^2/2 - (n-2)|\nabla u|^2/2 - nG(|u|)\right)dx \\ &+ \int_{m<r<2m+t}(\ln(m)/\ln(r))\left(n|u_t|^2/2 - (n-2)|\nabla u|^2/2 - nG(|u|)\right)dx \\ &- \int_{m<r<2m+t}(\ln(m)/\ln^2(r))\left(|u_t|^2/2 + |\nabla u|^2/2 - G(|u|)\right)dx \\ &+ \int_{2m+t<r}\operatorname{Re} e_{1m}\bar{u}_r u_t + e_{2m}\left(|u_t|^2 - |\nabla u|^2 + 2G(|u|)\right)dx = 0,\end{aligned}$$

where $|e_{im}(r, t)| < C$ for $r > 2m + t$. Integrating with respect to t we obtain

(2.2)

$$\begin{aligned}&\int \operatorname{Re}\phi_m(r, t)\bar{u}_r u_t\,dx - \int \operatorname{Re}\phi_m(r, 0)\bar{u}_r(0)u_t(0)\,dx \\ &= \int_0^t\int_{r<m}\left(-n|u_t|^2/2 + (n-2)|\nabla u|^2/2 + nG(|u|)\right)dx\,ds \\ &+ \int_0^t\int_{m<r<2m+s}(\ln(m)/\ln(r))\left(-n|u_t|^2/2 + (n-2)|\nabla u|^2/2 + G(|u|)\right)dx\,ds \\ &+ \int_0^t\int_{m<r<2m+s}(\ln(m)/\ln^2(r))\left(|u_t|^2/2 + |\nabla u|^2/2 - G(|u|)\right)dx\,ds \\ &+ \int_0^t\int_{r>2m+s}\operatorname{Re}\left[e_{1m}\bar{u}_r u_t + e_{2m}\left(|u_t|^2 - |\nabla u|^2 + 2G(|u|)\right)\right]dx\,ds.\end{aligned}$$

Equation (2.2) is all that is needed to show instability of the ground state and in the appendix we will show that (2.2) holds for radial solutions as defined.

LEMMA 2.2. *R_1 and R_2 are invariant regions under the flow of* (0.1) *for the solutions that satisfy the energy inequality.*

PROOF. We will show this by contradiction. Let $(u_0, u_1) \in R_1$ and assume there is a t_0 such that $(u(t_0), u_t(t_0)) \notin R_1$. By lower semicontinuity of $K(u(t))$ there is a minimal t_1 such that $(u(t_1), u_t(t_1)) \notin R_1$, i.e. $K(u(t_1)) \leqslant 0$ and $K(u(t)) > 0$, $0 \leqslant t < t_1$. Now

$$\frac{1}{n}\int |\nabla u(t_1)|^2\,dx \leqslant \varliminf_{\substack{t\to t_1\\ t<t_1}} \frac{1}{n}\int |\nabla u(t)|^2\,dx$$
$$\leqslant \varliminf_{\substack{t\to t_1\\ t<t_1}} \left(\frac{1}{n}\int |\nabla u(t)|^2\,dx + \frac{1}{n}K(u(t))\right)$$

and therefore

$$\frac{1}{n}\int |\nabla u(t_1)|^2\,dx \leqslant \varliminf_{t\to t_1} J(u(t)) \leqslant \varliminf_{t\to t_1} E(u(t), u_t(t)) < J_0.$$

But we also have $K(u(t_1)) \leqslant 0$ and this contradicts the definition of J_0 as the inf of $(1/n)\int |\nabla u|^2\,dx$ restricted to $K(u) \leqslant 0$ and $u \neq 0$. Therefore R_1 is an invariant region. Similarly we can show that R_2 is also invariant.

THEOREM 2.3. *Consider solutions of* (0.1) *that are radially symmetric and satisfy the energy inequality with Cauchy data $(u_0, u_1) \in R_2$. Then either*

(i) *the solutions exist only locally in time $[0, T_0)$, $T_0 < \infty$, with $\|u(t_n)\| \to \infty$ as $t_n \to T_0$ for some sequence (t_n), or*

(ii) *the solutions exist globally $[0, \infty)$ and $\exists$ a sequence $t_n \to \infty$ such that $\|u(t_n)\| \to \infty$ as $t_n \to \infty$.*

In either case the ground state is unstable.

PROOF. Statement (i) is just a consequence of the local existence of the solution [**8**].

Now we show (ii) by contradiction. Assume that the solution remains bounded in $H_r^1(\mathbf{R}^n)$. Then there is an $\varepsilon > 0$ such that $K(u(t)) < -\varepsilon\ \forall t$ since otherwise we would have a sequence (t_k) such that (since R_2 is invariant) $K(u(t_k)) \to 0$.

$$\frac{1}{n}\int |\nabla u(t_k)|^2\,dx = J(u(t_k)) - \frac{1}{n}K(u(t_k)) < E(u_0, u_1) - \frac{1}{n}K(u(t_k))$$

and if we extract a weakly convergent subsequence $u(t_k) \overset{w}{\rightharpoonup} u^* \in H_r^1(\mathbf{R}^n)$, $u(t_k) \to u^* \in L_r^p$, $2 < p < 2 + 4/(n-2)$, we get

$$\frac{1}{n}\int |\nabla u^*|^2\,dx \leqslant E(u_0, u_1) < J_0$$

(since $(u_0, u_1) \in R_2$) and $K(u^*) \leqslant 0$. But this contradicts the definition of J_0. Therefore such an ε exists.

Now from Lemma 1.2 we have $|u(r,t)| \leqslant C_n\|u(t)\|r^{(1-n)/2} < C_n C r^{(1-n)/2}$ and this implies

$$G(|u(r,t)|) = \int_0^{|u(r,t)|} f(s)\,ds \geqslant 0 \quad \text{for } r \text{ large}.$$

Now the right-hand side of equation (2.2) can be bounded above as follows:

$$\int_{m<r<2m+t} \left(n\ln(m)/\ln(r) - \ln(m)/(\ln(r))^2\right)G(|u|)\,dx < n\int_{m<r} G(|u|)\,dx,$$

$$\int_{m<r<2m+t} \left((n-2)\ln(m)/(2\ln(r)) + \ln(m)/2(\ln(r))^2\right)|\nabla u|^2\,dx$$
$$< \frac{n-2}{2}\int_{m<r} |\nabla u|^2\,dx + \frac{1}{2\ln(m)}\int_{m<r} |\nabla u|^2\,dx,$$

$$\int_{r>2m+t} \left(|\bar{u}_r u_t| + |u_t|^2 + |\nabla u|^2 + |G(|u|)|\right)dx$$
$$\leqslant C_0\left(\int_{2m<r} |\nabla u_0|^2 + |u_1|^2 + G(|u_0|)\,dx\right)$$

since radial solutions are strong outside a light cone containing the origin and therefore can be estimated by the initial data at the base. Consequently (2.2) will be bounded above by

$$\int \operatorname{Re}\phi_m(r,t)\bar{u}_r(r,t)u_t(r,t)\,dx - \int \operatorname{Re}\phi_m(r,0)\bar{u}_r u_1\,dx$$
$$\leqslant \int_0^t \left(K(u(s)) + \frac{1}{2\ln(m)}\int_{m<r} |\nabla u|^2\,dx\right. \tag{2.3}$$
$$\left. + C_1\int_{2m<r} \left(|\nabla u_0|^2 + |u_0|^2 + |u_1|^2 + G(|u_0|)\right)dx\right)ds.$$

Choose m large enough so that

$$\frac{1}{2\ln(m)}\int |\nabla u|^2\,dx < \frac{\varepsilon}{4}, \quad C_1\int_{2m<r} \left(|\nabla u_0|^2 + |u_0|^2 + G(|u_0|)\right)dx < \frac{\varepsilon}{4}.$$

Since $K(u(t)) < -\varepsilon$ equation (2.3) becomes

$$\int \operatorname{Re}\phi_m \bar{u}_r u_t\,dx - \int \operatorname{Re}\phi_m(r,0)\bar{u}_{0r}u_1\,dx \leqslant -\varepsilon t/2 \tag{2.4}$$

but $|\phi_m(r,t)| \leqslant Ct/\ln(t)$ for t large. Then (2.4) implies that

$$|\nabla u(t)|_2 |u_t(t)|_2 > C(\varepsilon)\ln(t) \quad \text{for } t \text{ large}.$$

This contradicts the assumption that $u(t)$ is bounded in $H_r^1(\mathbf{R}^n)$. Therefore there is a sequence (t_n) such that $\|u(t_n)\| \to \infty$.

Finally to show that this implies instability we note that for $u_0(x) = u_e(x/\lambda)$, $\lambda > 1$, $u_1(x) = 0$, we have $(u_0, u_1) \in R_2$ and is close to the ground state.

REMARK. It is worthwhile noticing that if the initial data are in R_1 then there is a global solution with this initial data. To see this by the energy inequality we have

$$\frac{1}{2}\int |u_t|^2\,dx + \frac{1}{n}\int |\nabla u|^2\,dx + \frac{1}{n}K(u) < J_0$$

and since R_1 is invariant, then $K(u(t)) > 0\ \forall t$. Therefore

$$\frac{1}{2}\int |u_t|^2\,dx + \frac{1}{n}\int |\nabla u|^2\,dx \leqslant J_0 \quad \text{and} \quad \int G(|u(t)|)\,dx$$

are bounded and this implies that we can find a global solution.

Appendix. To show that (2.2) holds for weak radial solutions we will follow the method of Strauss [**11**] to prove that the solutions are continuous in t off the origin.

LEMMA A. *For every* $\delta > 0$ *the weak radial solutions* u *are*

$$u \in C\big([0,T),\, H_r^1(\mathbf{R}^n \setminus B_\delta)\big), \qquad u_t \in C\big([0,T),\, L_r^2(\mathbf{R}^n \setminus B_\delta)\big)$$

for all T *where the solutions exist* $[B_\delta = \{r,\, r < \delta\}]$.

PROOF. Let ρ_k be an approximate identity and $u_k(t) = \rho_k * u(t)$. Then $u_k(t) \to u(t)$ in $H_r^1(\mathbf{R}^n)$. Let $\chi_\delta(r)$ be a smooth function such that $\chi_\delta(r) = 0$, $r < \delta/2$, and $\chi_\delta(r) = 1$, $r > \delta$.

Convoluting equation (0.1) with ρ_k we obtain

$$u_{ktt} - \Delta u_k + \rho_k * f(|u|)\arg u = 0. \tag{A.1}$$

Multiply (A.1) by $\chi_\delta \bar{u}_{kt}$ to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \left(|u_{kt}|^2 + |\nabla u_k|^2\right)\chi_\delta\,dx \\ &\qquad - \int \operatorname{Re}\left(\chi_\delta\left(\rho_k * f(|u|)\right)\bar{u}_{kt} + \chi_{\delta r} u_{kr}\bar{u}_{kt}\right)dx = 0.\end{aligned} \tag{A.2}$$

Now since $u \in L^\infty([0,T), H^1(\mathbf{R}^n))$ and is radially symmetric, then by Lemma 1.1 $|u(r,t)| < c$ for $r > \delta$ and this implies that $\chi_\delta(\rho_k * f(|u|)) \to \chi_\delta f(|u|) \in L^\infty([0,T), L_r^2(\mathbf{R}^n))$ and passing to the limit as $k \to \infty$ in (A.2) we get the desired result.

LEMMA B. *Equation* (2.2) *holds for weak radial solutions.*

PROOF. As before we work with equation (A.1). Multiply (A.1) by $\psi_{\delta,m}(r,t)\bar{u}_{kr}(r,t)$ where

$$\psi_{\delta,m}(r,t) = \begin{cases} 0, & r < \delta, \\ (r-\delta)m/(m-\delta), & \delta < r < m, \\ \phi_m(r,t), & r > m, \end{cases}$$

$\phi_m(r, t)$ is defined in §2. Then we have

$$\text{(A.3)}\quad \begin{aligned} &\frac{d}{dt}\int \operatorname{Re}\psi_{\delta,m}\bar{u}_{kr}u_{kt}\,dx - \int \operatorname{Re}\psi_{\delta,mt}\bar{u}_{kr}u_{kt}\,dx \\ &+\int \left(r^{n-1}\psi_{\delta,m}\right)_r |u_{kt}|^2/2r^{n-1}\,dx \\ &-\int \left((n-1)\psi_{\delta,m}/r - \left(r^{n-1}\psi_{\delta,m}\right)_r/2r^{n-1}\right)|\nabla u_k|^2\,dx \\ &+\int \operatorname{Re}\rho_k * f(|u|)\arg u\psi_{\delta,m}\bar{u}_{kr}\,dx = 0. \end{aligned}$$

As before we integrate with respect to t and take the limit as $k \to \infty$ where each integral term converges to its proper value:

$$\begin{aligned} \int \rho_k * f(|u|)\arg u\psi_{\delta,m}\bar{u}_{kr}\,dx &\to \int f(|u|)\arg u\psi_{\delta,m}\bar{u}_r\,dx \\ &= -n\int_{\delta<r<m} m/(m-\delta)G(|u|)\,dx + \text{terms independent of } \delta. \end{aligned}$$

Let $\delta \to 0$. Then from the above equation and (A.3) we obtain equation (2.2).

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